Proc. R. Soc. Lond. A **382**, 295–306 (1982)
Printed in Great Britain

Mach's principle and the structure of dynamical theories

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(*Communicated by R. Penrose, F.R.S. – Received* 6 *March* 1981 – *Revised* 18 *February* 1982)

A structure of dynamical theories is proposed that implements Mach's ideas by being relational in its treatment of both motion and time. The resulting general dynamics, which is called *intrinsic dynamics* and by construction treats the evolution of the entire Universe, is shown to admit as special cases Newtonian dynamics and Lorentz-invariant field theory provided the angular momentum of the Universe is zero in the frame in which its momentum is zero. The formal structure of Einstein's general theory of relativity also fits the pattern of intrinsic dynamics and is Machian according to the criteria of this paper provided the so-called thin-sandwich conjecture is generically correct.

1. A framework for implementing Mach's ideas

Dynamics is generally presented as the theory of the motion of arbitrary dynamical systems in space and time. However, as we pointed out in our earlier papers (Barbour 1974, 1982; Barbour & Bertotti 1977), a fully relational (and hence Machian) theory should start by considering the relative motion of the Universe treated as a single entity and then recover the motion of subsystems within the background provided by the Universe at large. This leads to explicit determination of the inertial frames of reference in terms of which dynamics is usually formulated.

Let us consider how a Machian scheme will differ from the framework of classical dynamical theory (see, for example, Lanczos 1949; Synge 1960), in which the basic concept is the *configuration space* Q of the system under consideration. Its points are defined by a set of variables $q = (q_1, q_2, \ldots)$. It is sometimes convenient to regard the time t as an extra dynamical variable $q_0 = t$, and adjoin it to q, obtaining thereby the *space of events* QT. A history of the system is represented by a curve in QT, the points of which are labelled by an arbitrary and monotonic parameter λ, the label time. This history is governed by an action $S = \int \mathrm{d}\lambda\, \mathscr{L}(q, q_\lambda)$ and the corresponding variational principle. The Lagrange function $\mathscr{L}$ is homogeneous of degree one in $q_\lambda = \mathrm{d}q/\mathrm{d}\lambda$, so that the action S is invariant under an arbitrary transformation of the parameter λ. Therefore $\mathscr{L}$ defines a Finsler metric $\mathrm{d}s = \mathrm{d}\lambda\,\mathscr{L}$ in QT and, if it is not degenerate, determines a unique history given its initial configuration q and its

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initial direction $\mathrm{d}q$. We propose to construct Machian theories in accordance with the same basic scheme but with two modifications.

First, the configuration q is generally defined in a *frame of reference*. Typically, a set of N point particles is determined by their $3N$ Cartesian coordinates $\boldsymbol{r}_i$ ($i = 1, 2, \ldots, N$); here $q = (\boldsymbol{r}_i)$. However, a classical (i.e. non-relativistic) system of N particles is really defined only by their relative distances r_{ij}. Because of the underlying Euclidean geometry, only $3N-6$ of these distances are independent. There is a six-parameter set of $q = (\boldsymbol{r}_i)$ that corresponds to a given set of r_{ij}; the sets are related to each other by a transformation of the *Euclidean symmetry group* E_0:

$$E_0\colon \boldsymbol{r} \to \boldsymbol{r}' = \mathbf{A}\cdot\boldsymbol{r} + \boldsymbol{h}, \tag{1.1}$$

where $\mathbf{A}$ is an orthogonal three-dimensional matrix and $\boldsymbol{h}$ is a vector. The configuration space Q is decomposed by E_0 into *orbits*, denoted by $\{q\}$, defined as the set of qs obtained by the application of all the elements of E_0. It is natural to regard the r_{ij} as primary, to consider the different representations q of a given $\{q\}$ as representing the same state of the system, to call the set of all orbits $\{q\}$ the *intrinsic* (or *relative*) *configuration space* (i.c.s.) of the system and to denote it by Q_0. In more complicated cases, for example in field theory, the introduction of explicit relative configuration variables like r_{ij} is not possible; the i.c.s. can be constructed only through the underlying group-theoretical structure, as will be explained later.

The Newtonian or Lorentzian forms of dynamics for general physical systems are formulated in Q. In contrast, we shall show here how, if the Universe satisfies certain conditions at infinity (in particular, if it is finite), its dynamics can be formulated in Q_0 and Newtonian or Lorentzian dynamics recovered in their conventional forms for isolated subsystems. The conflict between absolute and relative motion is then resolved. We shall say that a dynamical theory formulated in Q_0 implements the *first Mach principle*.

Our second modification concerns the treatment of time. For a system in QT, the time t is always provided by a clock exterior to the system. For such a t, the 'speed' at which the system moves through Q is well defined and motions along the *same* curve in Q correspond to different curves in QT. But if the points q of Q represent the entire Universe, it is hard to see what meaning could be attached to saying that absolutely everything is speeded up by the same amount: all the observable relations are still run through in the identical sequence. As was pointed out by Barbour (1974), the easiest way out of the dilemma is to assume that the Universe as a whole does not evolve in some Q_0T obtained by adjoining an independent time to Q_0, but simply in Q_0. In this view, the passage of time merely reflects the Universe's moving from one point of Q_0 to another. This is Leibniz's concept of time as merely the successive order of things: instants are defined by the successive relative configurations of the Universe (see Leibniz 1716). These define a curve in Q_0 whose points can be labelled by a monotonic and continuous parameter λ, a purely topological label with no metrical properties associated with it. We shall say that a theory that dispenses with an independent time and treats motion in Q_0 alone implements the *second Mach principle* (this expression was coined by Mittelstaedt (1976); Milne (1948) also advocated the construction of dynamical theory without recourse to time as an independent element).

As we pointed out in Barbour & Bertotti (1977), a Machian theory will be invariant under the *Leibniz group*:

$$E: \boldsymbol{r} \to \boldsymbol{r}' = \mathbf{A}(\lambda)\cdot \boldsymbol{r} + \boldsymbol{h}(\lambda), \tag{1.2a}$$

$$\lambda \to \lambda' = f(\lambda), \quad \mathrm{d}f/\mathrm{d}\lambda \neq 0, \tag{1.2b}$$

where the scalar f, the orthogonal matrix $\mathbf{A}$ and the vector $\boldsymbol{h}$ are arbitrary functions of the label λ.

In the first part of the paper we shall show how theories of the Universe as a whole invariant under (1.2), which involves seven arbitrary functions, can lead to theories of subsystems that are invariant only with respect to the much more restricted (finite-parameter) Galileo or Lorentz groups. This is achieved by a new – and Machian – form of dynamics, called *intrinsic dynamics*. In the final part we shall show that intrinsic dynamics has essentially the same structure as gauge theory and Einstein's general theory of relativity.

2. Intrinsic dynamics

The previous considerations strongly suggest that a Machian theory will be realized by a variational principle in the space of orbits Q_0, in analogy with the variational principles of classical mechanics in QT. Such a principle is given by a Finsler metric in Q_0. In Barbour & Bertotti (1977), we achieved this aim by writing down a Lagrange function directly in terms of the relative distances r_{ij} between point particles of masses m_i. While this was satisfactory in showing how a Newtonian-type dynamics invariant under only the Galileo group could arise locally as a good approximation from a Machian dynamics invariant globally under the Leibniz group (1.3), an unsatisfactory feature was the prediction of anisotropic effective masses, in contradiction with experiment. There also appeared to be no simple extension of such an approach to encompass field theory. Both the problems can be resolved in what we shall call *intrinsic dynamics*.

To define a distance in Q_0 a new mathematical tool has to be introduced, called the *intrinsic differential*. We shall see later that it is closely related to gauge theory. To grasp the idea, consider the example of a real scalar field ϕ defined on two-dimensional Euclidean space: $\phi = \phi(x, y)$. Suppose two successive 'photographs', called ϕ_1 and ϕ_2, are taken of such a field as it evolves. Each shows a pattern of intensities, the field strength ϕ at each point. The *relative* disposition of the intensities in ϕ_1 defines the position of ϕ_1 in Q_0. The descriptions in Q are obtained by laying a Cartesian grid in some arbitrary manner on the (x, y)-plane and determining the set of values $\phi_1(x, y)$, which fixes a q_1 in Q. The orbit $\{\phi_1\}$ is obtained by placing the Cartesian grid in all possible ways, and similarly for $\{q_2\}$. If ϕ_1 and ϕ_2 differ intrinsically, $\{q_1\}$ and $\{q_2\}$ are distinct.

If we now ask after the variation in ϕ between the two photographs, we come up against the problem that led Newton to introduce his concept of absolute space. Namely, we need to have some rule that enables us to fix the grid on ϕ_2 relative to the grid on ϕ_1 (which can always be chosen arbitrarily). Intrinsic dynamics is based on a simple principle which dispenses with a need for absolute space: namely, calculate first the difference $\mathrm{d}\phi = \phi_2 - \phi_1$ with arbitrary Cartesian grids and then evaluate the L_2 distance

$$\mathrm{d}s^2 = \int \mathrm{d}x \int \mathrm{d}y\, (\mathrm{d}\phi)^2 \tag{2.1}$$

(of course, we assume the required convergence properties); then by the action of E_0 shift one grid with respect to the other until $\mathrm{d}s$ is minimized. This procedure *stacks* ϕ_2 relative to ϕ_1 and its significance is clear: it reduces the difference between ϕ_1 and ϕ_2 to the smallest amount possible (as measured by (2.1)); the corresponding minimal $\mathrm{d}s$ can be called the *distance* between ϕ_1 and ϕ_2. It is coordinate-independent and determined globally.

In general, assume that a positive definite metric

$$\mathrm{d}s^2 = \langle \mathrm{d}q|\mathrm{d}q\rangle, \quad q_2 - q_1 = \mathrm{d}q$$

is defined in Q. Under the action of the Lie algebra of E_0,

$$q \to q' = q + \sum_\alpha \epsilon_\alpha O_\alpha q, \tag{2.2}$$

where O_α are the operators of infinitesimal translations and rotations. Since $\mathrm{d}q$ is infinitesimal, it is sufficient to act upon q_1 or q_2 with the algebra (2.2); we must then minimize the form

$$\langle \mathrm{d}q + \sum_\alpha \epsilon_\alpha O_\alpha q | \mathrm{d}q + \sum_\alpha \epsilon_\alpha O_\alpha q\rangle,$$

with respect to ϵ_α. This defines $\epsilon_{0\alpha}$ and a metric in Q_0:

$$\mathrm{d}s_0^2 = \langle \mathrm{d}q + \sum_\alpha \epsilon_{0\alpha} O_\alpha q | \mathrm{d}q + \sum_\alpha \epsilon_{0\alpha} O_\alpha q\rangle = \langle \mathrm{d}_\mathrm{I} q|\mathrm{d}_\mathrm{I} q\rangle.$$

The expression

$$\mathrm{d}_\mathrm{I} q = \mathrm{d}q + \sum_\alpha \epsilon_{0\alpha} O_\alpha q \tag{2.3}$$

is the *intrinsic differential* and, of course, is nothing but the part of $\mathrm{d}q$ that is orthogonal to the orbit of the group at q:

$$0 = \langle \mathrm{d}_\mathrm{I} q|O_\beta q\rangle = \langle \mathrm{d}q + \sum_\alpha \epsilon_{0\alpha} O_\alpha q|O_\beta q\rangle. \tag{2.4}$$

If we now consider a geodesic principle in Q determined by a Q-metric $\langle \mathrm{d}q|\mathrm{d}q\rangle$,

$$\delta S_Q = 0, \quad S_Q = \int \mathrm{d}\lambda\, \langle q_\lambda|q_\lambda\rangle^{\frac{1}{2}}, \tag{2.5}$$

we can construct the related action

$$S = \int \mathrm{d}\lambda\, \langle q_\lambda + \sum_\alpha a_\alpha(\lambda)\, O_\alpha \phi | q_\lambda + \sum_\alpha a_\alpha(\lambda)\, O_\alpha q\rangle^{\frac{1}{2}}. \tag{2.6}$$

The minimization of S with respect to a_α leads to the functions $a_{0\alpha}(\lambda)$ (for any given history $q(\lambda)$) corresponding to the intrinsic differential

$$\mathrm{d}_\mathrm{I} q = (q_\lambda + \sum_\alpha a_{0\alpha}(\lambda)\, O_\alpha q)\, \mathrm{d}\lambda,$$

and to an action and corresponding variational principle

$$\delta S = 0, \quad S = \int \langle \mathrm{d_I} q | \mathrm{d_I} q \rangle^{\frac{1}{2}}. \tag{2.7}$$

The action (2.6) contains the true dynamical variables q and the a_α, which can be regarded as auxiliary variables or, better, since the λ-derivatives of the $a_\alpha(\lambda)$ do not occur in (2.6), as constraint variables. The constraints corresponding to the a_α do not reflect the application of real forces to the system, but rather are introduced to obtain a variational principle that holds in an arbitrary frame of reference; they are therefore what Dirac (1964) calls *primary first-class constraints*. Whenever one has such a variational principle, neither the true dynamical variables (the qs) nor the constraint variables (the a_α) are uniquely determined by the equations of motion that follow from the variational principle. The arbitrariness in the solution (for given initial conditions) corresponds to the possibility of representing the same physical solution in all possible frames of reference. Thus, the physical solution is unique but the description of it is not.

Suppose an arbitrary path is defined in Q_0. It can be represented in Q by many different paths; however, for any given initial q in Q there is a unique representative path in Q distinguished by the fact that it cuts all the orbits in Q orthogonally. This is the path found by the stacking procedure outlined at the start of this section; we shall call it the stacked path. Note that along the stacked path the a_α in (2.6) vanish.

The group E_0 generates in the configuration space Q a group of motions that leave the metric $\langle \mathrm{d}q | \mathrm{d}q \rangle$ invariant. The trajectories determined by the action S_Q in (2.5) are the geodesics of this metric.

THEOREM. *The physically distinct solutions to the Q_0-problem* (2.7) *are the geodesics of the problem* (2.5) *that cut the orbits orthogonally.*

This theorem is well known from the theory of motions in Riemannian spaces (see, for example, Eisenhart 1933), but it is worth giving a separate proof.

Proof. The action for (2.7) along any virtual path in Q_0 is independent of the path by which it is represented in Q. Therefore, calculate it on the stacked representative path in Q. But on this path $S = S_Q$, since then the $a_\alpha = 0$. If a stacked path satisfies the principle (2.5), it is extremal in the set of all virtual paths in Q, including the stacked virtual paths. But it then satisfies the principle (2.7) as well, because for (2.7) it is sufficient to compare a realized path with stacked virtual paths, on which the two actions are equal. Hence any stacked geodesic of the principle (2.5) is a solution of (2.7). Moreover, this exhausts all the physically distinct solutions to (2.7). For suppose any initial condition for (2.7) is specified by some q and $\mathrm{d}q$. By an allowed transformation, we can also make $\mathrm{d}q$ perpendicular to the orbit $\{q\}$ of q (stacking of the initial condition). Since we assume $\langle \mathrm{d}q | \mathrm{d}q \rangle$ non-degenerate, there is a unique solution to the problem (2.5) with this initial condition. Now (2.5) is still invariant under the six-parameter Euclidean group E_0, so that by Noether's theorem, part I, any solution to the problem (2.5) conserves the quantities

$$P_\alpha = \langle q_\lambda | q_\lambda \rangle^{-\frac{1}{2}} \langle q_\lambda | O_\alpha q \rangle. \tag{2.8}$$

But the vanishing of the P_α is precisely the condition (2.4) which ensures that a particular path in Q is stacked. Thus, an initial condition for (2.5) that is stacked generates a solution that is stacked at all times. In geometrical language, if a geodesic cuts an orbit of Q orthogonally at the initial instant, it will cut all subsequent orbits orthogonally as well.

The P_α in (2.8) are the total momentum and angular momentum of the Universe associated with the action (2.5). Thus, the end effect of our intrinsic principle is to select the solutions of (2.5) that have vanishing momentum and angular momentum. This result is due to the implementation of the first Mach principle by the action (2.7). We mention in this connection the striking fact that the Universe does not appear to have any appreciable angular momentum, in agreement with the prediction of intrinsic dynamics.

3. Recovery of Newtonian mechanics in a Machian scheme

We shall now show how a particularly simple form of intrinsic dynamics leads to the Newtonian mechanics of a Universe of N gravitating point particles in Euclidean space. Let the particles have masses m_i, $i = 1, \ldots, N$, and let $q = (\boldsymbol{r}_i)$. Then a metric in Q is given by

$$\langle \mathrm{d}q | \mathrm{d}q \rangle = \sum_i m_i \mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_i. \tag{3.1}$$

This metric, which is *flat*, is too simple to yield non-trivial dynamics, which can be obtained by multiplying (3.1) by the 'conformal factor' $V(q) = \Sigma_{i<j} m_i m_j / r_{ij}$. Let us therefore consider the Q_0 variational principle

$$\delta I_{Q_0} = 0, \quad I_{Q_0} = \int \mathrm{d}\lambda \, [\langle q_\lambda + \sum_\alpha a_\alpha(\lambda) \, O_\alpha q | q_\lambda + \sum_\alpha a_\alpha(\lambda) \, O_\alpha q \rangle \, V(q)]^{\frac{1}{2}}. \tag{3.2}$$

By our theorem, we know that the physically distinct solutions to this Q_0 variational principle are the solutions to the related Q-problem.

$$\delta I = 0, \quad I = \int \mathrm{d}\lambda \, [\langle q_\lambda | q_\lambda \rangle \, V(q)]^{\frac{1}{2}},$$

for which the momentum and angular momentum vanish simultaneously. Defining

$$T = \sum_i m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}\lambda} = \langle q_\lambda | q_\lambda \rangle,$$

we obtain the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{V^{\frac{1}{2}}}{T^{\frac{1}{2}}} \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}\lambda}\right) = \frac{1}{2} \frac{T^{\frac{1}{2}}}{V^{\frac{1}{2}}} \frac{\partial V}{\partial \boldsymbol{r}_i}. \tag{3.3}$$

There is a unique choice of the arbitrary label λ that casts (3.3) in an especially simple form, namely when λ is chosen such that

$$T^{\frac{1}{2}} = V^{\frac{1}{2}}. \tag{3.4}$$

Then, if we denote the derivative with respect to the distinguished time label by a dot, (3.3) becomes

$$m \ddot{\boldsymbol{r}}_i = \tfrac{1}{2} \partial V / \partial \boldsymbol{r}_i,$$

which is identical to Newton's second law for gravitating point particles with gravitational constant $\frac{1}{2}$. Since we are not considering any other forces, this particular numerical value has no significance. The physically significant result is the condition (3.4), which tells us that the total energy of the system is exactly zero.

The results of this and the previous section show that there is no conflict between a fully relational formulation of the dynamical law of the Universe and the existence of special frames of reference and a special time variable that cast the laws of motion of subsystems of the Universe into particularly simple forms. The inertial frames of Newtonian theory arise from the fully Machian theory (3.2) when we perform the purely kinematic operation of stacking. The particles then obey Newton's laws as expressed in inertial frames, though the inertial frames have no absolute significance and are determined through the stacking procedure by the distribution and relative motion of the matter in the Universe. Similarly, absolute time does not exist in the scheme but rather is constructed from the Machian dynamics and the arbitrary Leibnizian time label λ by choosing it to enforce a specific (and rather obvious) simplicity requirement.

It is to be noted that the prediction of vanishing total energy of the Universe is a consequence of the particular and simple form of the Lagrange function and not of intrinsic dynamics (or Leibniz invariance) *per se*, as was pointed out to us by Kuchař. Consider the classical system with additive Lagrange function

$$\mathscr{L}_{\mathrm{cl}} = T + V = \langle \dot{q}|\dot{q}\rangle + V(q), \tag{3.5}$$

where $\langle \mathrm{d}q|\mathrm{d}q\rangle$ is a metric in Q and $-V(q)$ is the potential energy. If $W = T - V$ is the (conserved) total energy, the corresponding variational principle can be formulated in a way in which the time variable does not appear by using Jacobi's variational principle for the path of the system in Q:

$$0 = \delta \int \mathrm{d}\lambda \{(W + V(q))\,\langle \mathrm{d}q|\mathrm{d}q\rangle\}^{\frac{1}{2}} \tag{3.6}$$

(see, for example, Lanczos 1949). By replacing $\mathrm{d}q$ with the intrinsic differential $\mathrm{d}_{\mathrm{I}}q$ as in (2.3), one can then formulate the dynamics in a Machian form. Note that W is to be regarded as a fixed constant. If, as in the example above, the potential energy is negative, W can take the value zero, and we recover our principle (3.1). The possible appearance of the arbitrary constant W is a weakness of the theory and suggests we need a principle stronger than the invariance under (1.2*b*) (see later).

It has also been pointed out to us by Kuchař that, if the concept of the intrinsic derivative is abandoned, it may be possible to formulate classical dynamics in Q_0 without the requirement that the other constants of the motion vanish. This is the case when one can express the Lagrange function in a form independent of the coordinates on each orbit. They are then ignorable, and the classical procedure for their elimination can be followed (see, for example, Whittaker 1937). A well known example is provided by the two-body central-force problem, which can be described

(in Q_0T) by an action principle with the Lagrange function $\mathscr{L} = \frac{1}{2}m\dot{R}^2 - M^2/2mR^2 - V(R)$, where m is the reduced mass, M is the angular momentum, $V(R)$ is the potential energy, and R is the relative distance between the two bodies. This gives us a relational dynamics, but at the price of the introduction of the arbitrary constant M.

A very interesting discussion of the status of such constants when the complete Universe (assumed finite) is under consideration was given by Poincaré (1905, pp. 76–79 and ch. VIII) (see also Zanstra 1924). Are they to be regarded as 'accidental' (reflecting initial conditions for Galileo-invariant mechanics) or 'fundamental' (being integral parts of a relational dynamics of the Universe as a whole)? If the latter interpretation can be adopted, then, as Poincaré points out, there is no problem of absolute motion at all. Poincaré adduces some arguments against the latter interpretation, based mainly on the curious and arbitrary structure of the dynamical theories containing such constants.

In particular, he finds it curious that the evolution of the Universe is nearly but not quite determined by specification of the *observable* initial conditions, namely the initial relative positions and the initial relative velocities, but requires in addition specification of the above arbitrary constants. As ideal, Poincaré puts forward the following statement, which we shall call Poincaré's principle (for the strengthening of this principle to take into account the unobservability of absolute time, see Barbour (1982)):

> 'For the mind to be fully satisfied, the law of relativity would have to be enunciated as follows: The state of bodies and their mutual distances at any given moment, as well as the velocities with which these distances are changing at the moment, will depend only on the state of those bodies, on their mutual distances at the initial time, and on the velocities with which these distances were changing at the initial time.'

There are two points we should like to make in this connection. (i) The difficulty with the arbitrary structure disappears if the constants have vanishing values. In such a case, one obtains Leibniz-invariant laws of motion that have a simple and natural structure from the point of view of Q_0, and all conflict between absolute and relative motion disappears. In particular, the angular momentum must be zero in a theory based on the intrinsic differential. (ii) We shall see in the next section that general relativity can be cast in the form of a Q_0-theory with a structure that is a very natural generalization of (3.2).

Before turning to general relativity, we note that the same scheme of (3.5) and (3.6) can be used in field theory and leads to Lorentz invariance. For example, for a real scalar field $\phi(\boldsymbol{r})$ we can, from the classical kinetic energy $T = \int \mathrm{d}^3r\, \phi_t^2$, and potential energy $-V = \int \mathrm{d}^3r\, (\nabla\phi)^2$, construct the Leibniz-invariant action principle

$$\delta S = 0, \quad S = \int \mathrm{d}\lambda \left\{\left[W - \int \mathrm{d}^3r\,(\nabla\phi)^2\right] \int \mathrm{d}^3r\,(\phi_\lambda + \sum_\alpha a_\alpha O_\alpha \phi)^2\right\}^{\frac{1}{2}}.$$

This leads then to the solutions of the wave equation with vanishing total angular momentum in the frame in which the momentum $\boldsymbol{P}$ vanishes. The energy W must be greater than zero because the wave equation, being equivalent to a set of harmonic oscillators, has positive definite energy. Note, however, that the condition $P = 0$, $W \neq 0$ for the whole Universe is not Lorentz invariant.

The examples considered so far satisfy the restricted principle of relativity (Galilean invariance in Newtonian mechanics, Lorentz invariance in field theory). It is easy to show by counter examples that this is not a consequence of our Machian requirements but of the particular structure of the metric defined on Q.

4. Intrinsic dynamics, the gauge principle, and geometrodynamics

In this final section, we want to show that gauge theories and Einstein's general theory of relativity are examples of intrinsic dynamics. We begin with the electromagnetic field.

Suppose that Q is defined by the configurations of a three-dimensional vector A on Euclidean space; to the Euclidean symmetry group E_0 we adjoin the three-dimensional gauge group

$$\mathscr{H}_0\colon A \to A' = A + \nabla\Lambda, \tag{4.1}$$

where $\Lambda(\boldsymbol{r})$ is an arbitrary scalar function. We now introduce the intrinsic configuration space Q_0, whose points are the orbits of Q with respect to the infinite-dimensional group made up of E_0 and $\mathscr{H}_0$. The appropriate (flat) metric is

$$\langle \mathrm{d}A|\mathrm{d}A\rangle_F = \int \mathrm{d}^3r\,|\mathrm{d}A|^2. \tag{4.2}$$

In complete analogy with our basic scheme, a Q_0 variational principle is defined by the action

$$S = \int \mathrm{d}\lambda \left[\int \mathrm{d}^3r\,(A_\lambda + \nabla\Lambda + \sum_\alpha a_\alpha O_\alpha A)^2 \left(W - \int \mathrm{d}^3r\,|\nabla\times A|^2\right)\right]^{\frac{1}{2}}, \tag{4.3}$$

in which Λ and a_α are the auxiliary constraint variables and W is the total energy. The physically distinct solutions of (4.3) in the privileged time (defined by the condition analogous to (3.4)) are the solutions with vanishing momentum and angular momentum of the variational principle with the Lagrange function

$$L = \int \mathrm{d}^3r\,[(A_t + \nabla\Lambda)\cdot(A_t + \nabla\Lambda) - |\nabla\times A|^2]. \tag{4.4}$$

But setting $\Lambda = -A_0$ and identifying A_0 with the scalar potential in Maxwell's electrodynamics, we see that (4.4) is the Lagrange function for the free electromagnetic field. Thus, our intrinsic variation with respect to Λ is identical to the variation with respect to the scalar potential in Maxwell's theory. (Note that the stacking condition with respect to the gauge group is $\operatorname{div} A_t = 0$. By making a transformation belonging to $\mathscr{H}_0$, one can always achieve $\operatorname{div} A = 0$ for all times, in agreement with the fact that in the stacked form $\Lambda = A_0 = 0$.) Of course, it has long been known that A_0 plays the part of an auxiliary constraint variable and not that of a genuine dynamical variable (see, for example, Wheeler 1966). What we want to point out is that Newtonian or Lorentzian dynamics can be made to satisfy Poincaré's principle in exactly the same way that electrodynamics is gauge invariant.

The close parallel noted above between the gauge principle and intrinsic dynamics is repeated in general relativity (for simplicity, we consider the matter-free case, i.e. pure geometrodynamics): namely one can take as basic dynamical variable the metric tensor g_{ij} $(i,j = 1,2,3)$ of a three-dimensional Riemannian space and consider dynamical theories of the evolution of g_{ij}. The configuration space Q for such theories is obtained by regarding the ten functions g_{ij} of x^k as defining the points of Q. However, since a given g_{ij} can be referred to entirely arbitrary coordinate systems, all g_{ij} that can be obtained from one another by mere coordinate transformation must be regarded as identical and forming the orbit $\{g_{ij}\}$ of g_{ij}. Each such orbit constitutes a point of Q_0. For g_{ij} defined on a compact three-dimensional manifold, the corresponding Q_0 is known as *superspace* (see DeWitt 1970). The infinitesimal transformations generated by an arbitrary infinitesimal coordinate transformation are

$$\mathscr{E}_0\colon g_{ij} \to g'_{ij} = g_{ij} + \xi_{i;j} + \xi_{j;i}, \tag{4.5}$$

where ξ_i is an arbitrary three-vector field and the semicolon denotes the covariant derivative with respect to g_{ij} itself. The transformations (4.5) are a generalization of the gauge group (4.1) though an important difference here is that (4.5) represents the generalizations of both E_0 and $\mathscr{H}_0$. Moreover, the infinitesimal transformations (4.5) are not linear in g_{ij}.

The action principle for general relativity is usually expressed directly in terms of the four-dimensional metric $g_{\mu\nu}$ corresponding to Einstein's original discovery of the theory as a theory of space–time. It can, however, be recast as a theory of the dynamical evolution of Riemannian three-geometries g_{ij}. For this it is necessary to slice the four-dimensional space–time by arbitrary three-dimensional hypersurfaces and introduce the lapse N and shift N^i. Both are functions of position on the considered hypersurface, and the lapse tells one the orthogonal distance on the considered hypersurface in space–time to the next hypersurface of the slicing, while the shift tells one the connection between the coordinate systems used on the successive three-surfaces, i.e. N^i is the spatial displacement of the point on the next hypersurface with the same numerical values of the coordinates.

Then the Hilbert action for matter-free general relativity becomes

$$S[g_{ij}, N, N^i] = \int \mathrm{d}\lambda \int \mathrm{d}^3x\, N g^{\frac{1}{2}}\{(k_{ij}k^{ij} - k^i{}_i k^j{}_j)/4N^2 + R\}, \tag{4.6}$$

where $k_{ij} = \mathrm{d}g_{ij}/\mathrm{d}\lambda - 2N_{(i;j)}$, $N_{(i;j)} = \frac{1}{2}(N_{i;j} + N_{j;i})$, R is the Riemannian curvature of the hypersurface, $g = \det\|g_{ij}\|$, and the indices are raised and lowered with g_{ij} itself.

Varying with respect to N, we obtain

$$N = \tfrac{1}{2}R^{-\frac{1}{2}}(k_{ij}k^{ij} - k^2)^{\frac{1}{2}}. \tag{4.7}$$

Provided N is real and also $N \neq 0, \infty$ it is permissible to eliminate N from (4.6) by substituting (4.7). Introducing at the same time DeWitt's metric, $G^{ijkl} = \frac{1}{2}g\{g^{il}g^{kj} + g^{ik}g^{lj} - 2g^{ij}g^{kl}\}$, we cast (4.6) in the form

$$S[g_{ij}, N^k] = \int \mathrm{d}t \int \mathrm{d}^3x\,\{R[g]\,G^{ijkl}[g]\,(\dot g_{ij} - 2N_{(i;j)})\,(\dot g_{kl} - 2N_{(k;l)})\}^{\frac{1}{2}}. \tag{4.8}$$

This form of the action for geometrodynamics was first given by Baierlein *et al.* (1962). It satisfies our Machian principles, since N_i is a direct generalization of the constraint variables introduced in the general theory to implement the first Mach principle and, moreover, (4.8) is homogeneous of degree one in the time derivatives (and hence implements the second Mach principle).

Note, however, that (4.8) is much more complicated than the principles considered earlier, which were all based on a Riemannian metric in Q_0 (square root of a sum of squares), and, in fact, a conformally flat Riemannian metric. In contrast, in (4.8) the metric is a sum (integral) of square roots, so that (as Kuchař has pointed out to us) it is a Finsler metric. In this case, there is no scalar product and the ordinary definition of orthogonality does not apply. However, we can say that a given $\mathrm{d}g_{ij}/\mathrm{d}\lambda$ is orthogonal to the orbits of $\mathscr{E}_0$ if S in (4.8) is stationary with respect to N_i.

It should also be emphasized that general relativity is a very special theory among all possible theories of the dynamical evolution of Riemannian three-geometries in accordance with Poincaré's principle. As was shown by Hojman *et al.* (1976), the action (4.8) is almost uniquely determined by the requirement that the evolving three-geometries can be stacked to make a *four-dimensional* Riemannian space, i.e. space–time. (Our scheme based on the more restricted group $\mathscr{E}_0$ is more general and could therefore provide a framework to study theoretically violations of general covariance, in particular Lorentz invariance.)

In the light of our general approach to the problem of motion, the important thing about (4.8) is that, formally at least, it gives complete expression to Poincaré's principle, that is the future evolution of the system is determined in accordance with the basic law by specification of the initial position in Q_0 (g_{ij}) and the initial 'direction' $(\partial g_{ij}/\partial\lambda)$ at that point. The difference between the formal structure of general relativity and Newtonian mechanics is reflected in the fact that the latter can be represented in a form analogous to (4.8) only for the solutions with vanishing energy and angular momentum, whereas no such restriction must be made for general relativity: it is already in intrinsic form.

However, two reservations must be made. First, for an infinite Universe, the principle (4.8) must be augmented by boundary conditions at spatial infinity, which are quite alien to our general scheme. In fact, Q_0 cannot be meaningfully defined unless the Universe is finite in an appropriate sense and preferably closed. Such an assumption has been implicit in our entire work and we regard it as a necessary condition for satisfactory implementation of the idea that inertial motion is governed by the Universe as a whole. The second reservation relates to the fact that even for a closed Universe it is not known whether the Cauchy problem corresponding to (4.8) is generically uniquely solvable. This is essentially the thin-sandwich problem, first formulated by Baierlein *et al.* (1962), which is not yet solved though a uniqueness proof under certain conditions has been given by Belasco & Ohanian (1969).

Revising the opinion we expressed at the end of Barbour & Bertotti (1977), we

would say therefore that in its basic structure general relativity is Machian and gives expression to Poincaré's principle as a theory describing the evolution of closed three-geometries from intrinsically specified initial data. Though reached from a rather different and more general point of view, this is essentially the position of Wheeler (1966) (for matter-free general relativity, i.e. pure geometrodynamics; when matter is present, the situation is more complicated and must be considered separately).

We are both grateful to Karel Kuchař for several very helpful comments and numerous discussions which considerably clarified our work. We should also like to thank the referee for drawing our attention to Utiyama's (1956) paper, which is similar to ours in showing how a theory with a particular invariance can be transformed into another theory with a larger invariance group. Many mathematical relations are common to Utiyama's paper and ours though there are differences in the basic aims and the methods used to achieve them. Finally, J.B.B. is again indebted to Professor M. Fraccaro, Rettore of Collegio Cairoli, for hospitality at Collegio Cairoli during several visits to Pavia between December 1978 and June 1981; B.B. was supported by the Consiglio Nazionale delle Ricerche.

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